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BY

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ABSTRACT. A continuity property is proved for variable simply connected domains with locally connected boundaries. This theorem provides a link between limits of conformal mappings and of retractions. Applications are given to the space of retractions of a compact 2-manifold M^2, where it is shown that the space of deformations retractions is contractible and the space of nullhomotopic retractions has the same homotopy type as M^2. Other applications include a proof that the space of retracts of M^2 (with a natural quotient topology) is an absolute neighborhood retract, and a type of global solution to the Dirichlet problem.

0. **Introduction.** This article contains proofs for the results announced in [23]. The basic tool for this work is a continuity property for variable simply connected domains with locally connected boundaries (Theorem 2.2), which provides a link between limits of conformal mappings and of retractions. Preliminary material on conformal mapping theory appears in §1, including a continuity property for a fixed domain (Theorem 1.1) similar to Theorem 2.2.

In §§3 and 4, we use Theorem 2.2 to analyze two components of the space of retractions of an arbitrary compact 2-manifold M^2. In §3, we construct a contraction of the space of deformation retractions of M^2, while in §4 we show that the space of nullhomotopic retractions of M^2 has the same homotopy type as M^2. The methods of §4 are similar to those in [21] and [22]. In both §§3 and 4, the construction is essentially that first given by Alexander [2], [11, p. 524], as modified for retractions by Borsuk [4] and the author (Remark 3.2). Only in five cases (the 2-sphere, disk, annulus, projective

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plane and Möbius strip) does this work provide a complete analysis of the global properties of the space of retractions. For example, even in cases as simple as the torus or the disk with three holes, the space of retractions has a complicated infinite array of components [24].

In §5, we give the set of retracts of M^2 the quotient topology determined by the projection from the space of retractions, and prove that the space of retracts is an absolute neighborhood retract (Theorem 5.5). We also give a version for retracts of the continuity property (Theorem 5.4).

Finally, in §6 we present two global forms of the continuity property: a canonical form of the Schoenflies theorem and a type of simultaneous solution to the Dirichlet problem.

Part of the material in §3 appeared in the author's doctoral dissertation at the University of Illinois, Urbana-Champaign. The author would like to thank his advisor, Mary-Elizabeth Hamstrom, and the referee of [21] for a number of very useful ideas. He would also like to thank C. W. Neville and J. M. Gray for encouragement.

1. **Preliminaries on conformal mapping.** In this section we introduce concepts from conformal mapping theory which will be used in §2. As basic references for this section, see [1], [10], and [15]. Throughout this paper, B^2 will denote the closed unit ball in the Euclidean plane E^2, and C_α will denote the circle with center at the origin and radius α. The symbol $A(J, K)$ denotes the closed annular region bounded by simple closed curves J and K, where J lies in the bounded complementary domain of K. In particular, A^2 is used for $A(C_1, C_2)$. The abbreviations bdry, cl, int, im, diam, and dist are used respectively, for boundary, closure, interior, image, diameter, and distance.

Let G be a bounded simply connected domain in E^2. A point $v \in F = \text{bdry}(G)$ is *accessible from* G if there is an embedding $e\colon [0, 1] \to G \cup F$ such that $e([0, 1)) \subset G$ and $e(1) = v$. An *accessible point* of F consists of a point $v \in F$ and an equivalence class of embeddings, where we take e_1 equivalent to e_2 if and only if there is an embedding e as above which has image points in common with e_1 and e_2 arbitrarily close to v. By an abuse of language, we shall refer to v or to (v, e) as an accessible point.

Given distinct accessible points (v_1, e_1) and (v_2, e_2) of F, it is easy to find a mapping $E\colon [-1, 1] \to G \cup F$ such that

$$E((-1, 1)) \subset G, \quad E(-1) = v_1, \quad E(1) = v_2,$$

and, for some $\epsilon > 0$,

$$E(-t) = e_1(t) \quad \text{and} \quad E(t) = e_2(t),$$

for all $t \in [1 - \epsilon, 1]$. The set $E([-1, 1])$ is called a *cross cut* and splits G

into two components. A closed *H-interval* of F is the collection of accessible points of F which are accessible from one of the above two components [15, p. 59]. Thus E splits the accessible points of F into two H-intervals with only the *endpoints* (v_1, e_1) and (v_2, e_2) in common.

A decreasing sequence $\{H_n\}$ of H-intervals containing at most one accessible point is said to determine a *prime end* of G. There is a corresponding decreasing sequence $\{G_n\}$ of subdomains of G such that the accessible points of F accessible from G_n are exactly those in H_n and such that $P = \bigcap \mathrm{cl}(G_n)$ is contained in F. The set P depends only on the prime end determined by $\{H_n\}$, and we say that P *belongs* to this prime end. If the set which belongs to a prime end consists of one accessible point, we say the prime end is *of the first kind.*

The following known theorem is reminiscent of the result in [18, p. 262], where conditions are given equivalent to the existence of a conformal homeomorphism $f\colon \mathrm{int}(B^2) \longrightarrow G$.

1.1. THEOREM (CONTINUITY PROPERTY FOR A FIXED DOMAIN). *Let G be a bounded simply connected domain with boundary F, and let D be a closed disk contained in G. The following are equivalent.*

(a) *The boundary F is locally connected.*

(b) *The prime ends of G are all of the first kind.*

(c) *There is a continuous surjection $f\colon B^2 \longrightarrow G \cup F$ which is a conformal homeomorphism on $\mathrm{int}(B^2)$.*

(d) *There is a retraction φ of $E^2 \backslash \mathrm{int}(D)$ with image $E^2 \backslash G$.*

(e) *There is a canonical retraction ψ of $E^2 \backslash \mathrm{int}(D)$ with image $E^2 \backslash G$ mapping $G \backslash \mathrm{int}(D)$ onto F.*

1.2. REMARK. Compare this theorem with Theorems 2.2 and 5.4. See [26, p. 112] for three other topological conditions (one appearing in the proof below) equivalent to (a) through (e) above. The equivalence of (b) and (c) is an old result due to Carathéodory and others. A direct proof that (a) implies (d) was originally given by Borsuk [3]. In condition (e) above and in Theorem 2.2 (e), the word "canonical" means that, for any boundary F and any disk D in G, we are giving a procedure for constructing a unique retraction ψ with the desired properties.

PROOF. To prove that (a) implies (b), assume F is locally connected. Whyburn proves [26, p. 112] this is equivalent to saying each $v \in F$ is *accessible from all sides from* G, meaning that given any cross cut in G, v is accessible from either (or both) of the resulting components which has v on its boundary. It is then easy to see that (b) is true [15, p. 65]. The proof that (b) implies (c) can be found in [15, p. 67].

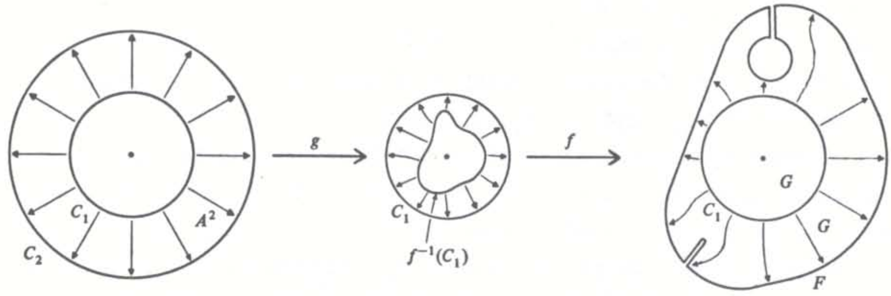

FIGURE 1

Next, we give a constructive proof that (c) implies (e). Without loss of generality we can assume that the closed disk D is the closed unit disk B^2. Let $g\colon A^2 = A(C_1, C_2) \longrightarrow A(f^{-1}(C_1), C_1)$ be a homeomorphism which is a radial contraction of a map conformal on the interior of the annulus. (See Figure 1.) Assuming we specify that f fixes the origin, the maps f and g and hence $h = f \circ g$ are only determined up to a rotation, but the construction which follows is independent of rotations. Let $\lambda\colon A^2 \longrightarrow C_2$ be the obvious radial retraction. We define a retraction ψ of $E^2 \setminus \operatorname{int}(B^2)$ onto $E^2 \setminus G$ by setting ψ equal to $h \circ \lambda \circ h^{-1}$ on $(G \cup F)\setminus \operatorname{int}(B^2)$ and ψ equal to the identity on $E^2 \setminus G$. The continuity of ψ only needs checking on F, and it is continuous there because h is continuous on C_2.

It is trivial that (e) implies (d), and assuming (d), a bounded portion of $E^2 \setminus G$ is a locally connected continuum [26, p. 26], and so its boundary F is also locally connected [26, p. 106]. Q.E.D.

For each nonnegative integer n, suppose there is given a bounded simply connected domain G_n in the plane such that some fixed closed disk D with a point u_0 as center is contained in all the G_n. Let $f_n\colon \operatorname{int}(B^2) \longrightarrow G_n$ be the unique conformal homeomorphism mapping the origin to u_0 and having positive derivative at the origin [15, p. 13].

1.3. DEFINITION. The domain G_0 is the *kernel* (*relative to* u_0) of the sequence $\{G_n : n \geqslant 1\}$ if G_0 is the largest domain containing u_0 such that each compact subset of G_0 is contained in G_n for sufficiently large n. (Alternatively, we can let G be the set of points with a neighborhood contained in the G_n for sufficiently large n. The component of G containing u_0 is the kernel.) The sequence $\{G_n\}$ *Carathéodory converges* (*relative to* u_0) to G_0 if every subsequence of $\{G_n\}$ has kernel G_0.

1.4. THEOREM (CARATHÉODORY). *The sequence* $\{G_n\}$ *Carathéodory converges to* G_0 *if and only if the sequence* $\{f_n\}$ *converges to* f_0 *uniformly on compact subsets of* $\operatorname{int}(B^2)$.

The same theorem is true for the functions f_n^{-1}. (See [15] for proofs and [8] for references.)

We shall also need versions of Theorems 1.1 and 1.4 for a general (second countable) 2-manifold M^2. Our technique will be to lift the simply connected domain G to the universal covering space of $M^2 \backslash \partial M^2$, which will be E^2 except for the cases where M^2 is the projective plane or the 2-sphere [1, p. 104]. Since G is simply connected and hence contractible in M^2, it is not hard to see that G will lift to homeomorphic copies in any covering space [19, pp. 66–68]. We shall start with a precompact simply connected domain G in M^2, and since we wish to obtain a bounded simply connected domain in E^2, we employ the following strategem: Attach a collar $\partial M^2 \times [0, 1]$ to ∂M^2, producing a 2-manifold N^2, and take the universal covering space of $N^2 \backslash \partial N^2$. If we suppose that M^2 has been given a conformal structure so as to make it a Riemann surface (a bordered Riemann surface if $\partial M^2 \neq \varnothing$ and including conjugates of conformal mappings if M^2 is not orientable [1]), then we can extend this to a conformal structure for N^2 as in [1, p. 118]. Finally, we can put a conformal structure on the universal covering space so as to make the covering projection conformal [1, p. 119].

If F, the boundary of G, consists of more than one point, then G is of hyperbolic type [1, pp. 141, 158, 204], i.e., conformally equivalent to $\text{int}(B^2)$. Let $\hat{G} \subset E^2$ be a component of $p^{-1}(G)$, where $p: E^2 \longrightarrow N^2 \backslash \partial N^2$ is the covering projection. While $\hat{F} = \text{bdry}(\hat{G})$ need not be homeomorphic to F, $\hat{F}$ will be locally connected if F is. Thus by Theorem 1.1(c), there is a continuous function $\hat{f}: B^2 \longrightarrow \hat{G} \cup \hat{F}$ which is a conformal homeomorphism on $\text{int}(B^2)$. The composition $f = p \circ \hat{f}$ gives a continuous surjection of B^2 onto $G \cup F$ which is conformal on $\text{int}(B^2)$. Hence we have the following.

1.5. THEOREM. *In an arbitrary (second countable) 2-manifold M^2, let G be a precompact simply connected domain with boundary F consisting of more than one point. Let D be a closed disk in G containing some $u_0 \in M^2$ in its interior. Then the five conditions of Theorem* 1.1 *make sense and are equivalent in this more general setting.*

For each nonnegative integer n, suppose G_n is a precompact simply connected domain in M^2 of hyperbolic type such that each G_n contains some fixed closed disk D with u_0 in its interior. Definition 1.3 may be used in the same form to define Carathéodory convergence of $\{G_n\}$ to G_0. For each n let f_n be a conformal homeomorphism of $\text{int}(B^2)$ onto G_n mapping the origin to u_0. Each f_n is uniquely determined up to a rotation, and a reflection if M^2 is not orientable. Thus we require that the images of the positive x-axis under the f_n all make the same angle at u_0, and, if M^2 is not orientable, we require that each f_n be directly conformal using a fixed orientation for D and $\text{int}(B^2)$.

1.6. THEOREM. *Using notation above, $\{G_n\}$ Carathéodory converges to G_0 if and only if $\{f_n\}$ converges to f_0 uniformly on compact subsets of* $\text{int}(B^2)$.

Finally, we state a lemma which is very useful in work with variable domains.

1.7. LEMMA (LINDELÖF [10, p. 33]). *Suppose an analytic function f is bounded on some bounded domain G (containing a point u_0) by a number $M > 0$, and suppose that for some $r > 0$, there is an arc of the circle $C = \{u\colon |u - u_0| = r\}$ disjoint from $\text{cl}(G)$, where the arc makes an angle of $2\pi/k$ for some integer k. Suppose, further, there is a number $m > 0$ such that for any boundary point w of G inside C, we have $\limsup_{u\to w} |f(u)| \leqslant m$. Then $|f(u_0)| \leqslant (mM^{k-1})^{1/k}$.*

2. **The continuity property for variable domains.** In this section we give an altered version of a theorem of D. Gaier [8, p. 395]. We have added three new equivalent conditions, but weakened the generality of the theorem. For other references, see [8] and [9, p. 27].

For each nonnegative integer n, let G_n be a bounded simply connected domain in E^2 such that each G_n has locally connected boundary F_n and contains some fixed closed disk D with a point u_0 as center. Let $f_n\colon B^2 \longrightarrow G_n \cup F_n$ be the function given by Theorem 1.1(c), which will be uniquely determined by requiring that f_n map the origin to u_0 and that f_n have positive derivative at the origin. Following [25, p. 337] and [8, p. 394], we give a definition which introduces a measure of the "smoothness" of the boundary F_n of our domain G_n.

2.1. DEFINITION. For every $\delta > 0$, consider all cross cuts Q in G_n missing u_0 and of diameter $\text{diam}(Q) \leqslant \delta$, and let $d(G_n\backslash Q)$ denote the diameter of the component of $G_n\backslash Q$ which does not contain u_0. Set

$$\eta_n(\delta) = \sup\{d(G_n\backslash Q)\colon \text{diam}(Q) \leqslant \delta\}.$$

One says that the sequence $\{F_n\colon n \geqslant 1\}$ *Fréchet converges* to F_0 if

(1) $\{G_n\}$ Carathéodory converges to G_0 (Definition 1.3), and

(2) $\lim_{n\to\infty,\delta\to 0} \eta_n(\delta) = 0$.

2.2. THEOREM (CONTINUITY PROPERTY FOR VARIABLE DOMAINS). *The following conditions are equivalent.*

(a) *The sequence* $\{F_n\}$ *Fréchet converges to* F_0.

(b) *Each accessible point* $v_0 \in F_0$ *is the limit of a sequence* $\{v_n \in F_n\}$ *of accessible points, and every subsequence of such a sequence has in turn a convergent subsequence.* (*See Remark* 2.3(1) *below.*)

(c) *The sequence* $\{f_n\}$ *converges to* f_0 *uniformly on the closed disk* B^2.

(d) *There are retractions* φ_n $(n \geqslant 0)$ *of* $E^2\backslash\text{int}(D)$ *with image* $E^2\backslash G_n$ *such that* $\{\varphi_n\}$ *converges to* φ_0 *uniformly on* $E^2\backslash\text{int}(D)$.

(e) *There are canonical retractions* ψ_n $(n \geqslant 0)$ *satisfying* (d) *such that each* ψ_n *maps* $G_n\backslash\text{int}(D)$ *onto* F_n.

2.3. REMARK. (1) Since each F_n is locally connected, all prime ends are of the first kind, and the notions of "prime end" and "accessible point" coincide (see Theorem 1.1). We say that a sequence $\{v_n \in F_n\}$ of accessible points *converges* to an accessible point $v_0 \in F_0$ if there are representing embeddings e_n for v_n $(n \geqslant 0)$ such that $\{e_n\}$ converges to e_0 uniformly on $[0, 1]$.

(2) It is not hard to see that condition (b) makes sense and is equivalent to (a) even in the more general case considered in [8, p. 395], where only F_0 is assumed locally connected and the convergence is assumed uniform only on $\text{int}(B^2)$. We choose to include condition (b) here because it may be of interest in its own right. For the proof of Theorem 2.2, we could partly refer to [8], but instead we shall outline an independent proof, relying on Lindelöf's lemma (Lemma 1.7) rather than the Dirichlet integral that is used in [8].

(3) We could increase the similarity between Theorems 1.1 and 2.2 by changing condition (a) of Theorem 1.1 to $\lim_{\delta\to 0}\eta_0(\delta) = 0$, which is clearly equivalent to the local connectedness of F_0 [8, p. 394]. Similarly, condition (b) of Theorem 1.1 could also be altered.

PROOF THAT (a) IMPLIES (b). Assume (a) is true and (b) is not true. The Carathéodory convergence implies the first condition of (b), so we can assume there is a sequence $\{v_n \in F_n : n \geqslant 1\}$ of prime ends with no convergent subsequence. Using the map f_0, we define *standard simple closed curves* $J(\alpha)$ and *standard radial arcs* $A(\gamma)$ in G_0 to be images under f_0 of circles C_α $(\alpha < 1)$, and of rays from the origin making an angle γ, respectively.

By taking a subsequence of $\{v_n\}$ (but keeping the same notation), we claim that we can choose a sequence $0 < \alpha_1 < \alpha_2 < \alpha_3 < \cdots$ converging to 1 and an $\epsilon > 0$ such that

(1) for all n, each point of $J(\alpha_n)$ can be joined to the corresponding point of F_0 by a portion of a standard arc of diameter less than $1/n$ lying except for its endpoints in G_0 but outside $J(\alpha_n)$,

(2) for all $n, J(\alpha_n)$ is a subset of G_n,

(3) for all n, each point of F_n is within β_n of F_0, where $\beta_n =$

$\operatorname{dist}(J(\alpha_n), F_0)$ (note that $\beta_n < 1/n$), and

(4) for all representing embeddings e_n of v_n such that $e_n(0) = u_0$, there is a terminal arc of $e_n([0, 1])$ outside $J(\alpha_n)$ having diameter greater than or equal to ϵ.

In condition (1) we use the fact that F_0 is locally connected, and in conditions (2) and (3) we take subsequences. If (4) were not true, we could construct a convergent subsequence of $\{v_n\}$ using the standard arcs $A(\gamma)$.

For the rest of this proof, assume that n is fixed and assume that e_n is a fixed representing embedding for v_n. For each angle γ, we use $A(\gamma)$ and e_n to attempt to construct a representing embedding e_γ for v_n as follows:

(5) Trace along $A(\gamma)$ to $J(\alpha_n)$ (using parameter values in $[0, 1 - \alpha_n]$ and the map f_0). Then, if possible, connect up with that point of $e_n([0, 1])$ outside $J(\alpha_n)$ with largest parameter value (denoted t_γ) using an arc of diameter less than or equal to $2/n$. Finally, finish the embedding along $e_n([0, 1])$. (See Figure 2.)

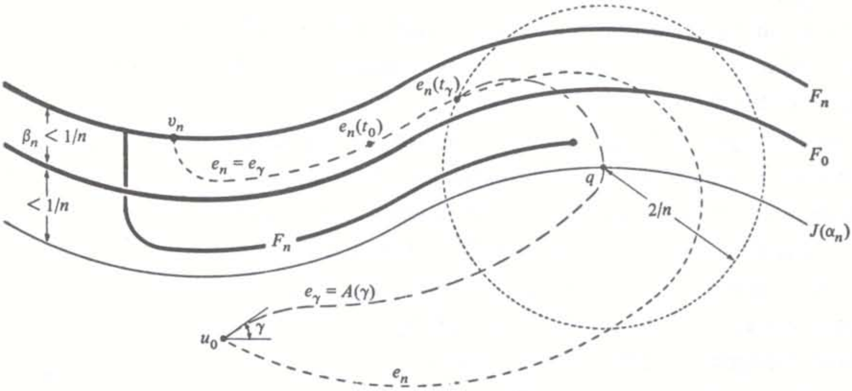

FIGURE 2

By (1), (2), and (3), e_γ and t_γ exist for some γ.

(6) Set $t_0 = \sup\{t_\gamma : 0 \leqslant \gamma < 2\pi\}$, and choose a fixed angle γ such that e_γ exists and such that $e_n([t_\gamma, t_0])$ has diameter less than $1/n$. For this fixed γ, denote the intersection of $A(\gamma)$ with $J(\alpha_n)$ by q.

(7) Choose points $q', q'' \in J(\alpha_n)$ such that $|q - q'|$, $|q - q''|$, and $|q' - q''|$ all lie between $5/n$ and $15/n$ and such that the arc $q'qq''$ of $J(\alpha_n)$ has diameter less than $50/n$. (Here and later in the proof, we might have to insist that n be sufficiently large.)

By (1) and (3), we can find points $s', s'' \in F_n$ and arcs $q's'$ and $q''s''$ of diameter less than $2/n$ whose interiors lie outside $J(\alpha_n)$ and inside G_n. By (5) and the construction of e_γ, the arcs $q's'$ and $q''s''$ do not intersect $e_\gamma([0, 1])$. Thus the arc $Q_n = s'q'qq''s''$ is a cross cut in G_n of diameter less than $60/n$. But the portion of $e_\gamma([0, 1])$ from q to v_n has diameter greater than or equal to ϵ. Thus for any $\delta > 0$, we can choose n large enough so that $\operatorname{diam}(Q_n) < \delta$. Hence

$$\eta_n(\delta) \geqslant d(G_n \backslash Q_n) \geqslant \epsilon > 0,$$

and so clearly condition (2) of Definition 2.1 does not hold.

PROOF THAT (b) IMPLIES (c). We shall sketch a proof modeled after that in [10, pp. 59–62]. We wish to show that $\{f_n : n \geqslant 1\}$ is equicontinuous on C_1 (= $\operatorname{bdry}(B^2)$). Suppose otherwise. Then by shifting to subsequences (but keeping the same notation), we can find distinct points r_n and s_n of C_1 ($n \geqslant 1$) such that the sequences $\{r_n\}$ and $\{s_n\}$ both converge to some $r_0 \in C_1$, and such that r_n and s_n correspond to distinct prime ends a_n and b_n of G_n, where $\{a_n\}$ and $\{b_n\}$ converge (as sequences of prime ends–see Remark 2.3) to distinct prime ends a_0 and b_0 of G_0. One of the arcs with endpoints r_n and s_n is constricted to r_0 as n tends to infinity, and denote the corresponding H-intervals in F_n by A_n. Denote the complementary H-intervals in F_n by B_n ($n \geqslant 1$).

(1) We can label the H-intervals determined by a_0 and b_0 with A_0 and B_0 so that the sequences $\{A_n\}$ and $\{B_n\}$ converge in the sense of condition (b) to A_0 and B_0, respectively. (In this and other proofs, each numbered statement is an assertion whose proof is clear or follows the statement.)

For the labeling, choose a prime end w_0 in F_0 not equal to either a_0 or b_0. Condition (b) shows that w_0 is the limit of a sequence $\{w_n\}$ of prime ends. By taking a subsequence, we can assume that $w_n \in A_n$ or $w_n \in B_n$ for all n. We label the H-interval containing w_0 with A_0 in the first case and with B_0 in the second case.

In order to show that condition (b) holds for the sequence $\{A_n\}$ and for A_0, let v_0 be any prime end of A_0. By condition (b) in the original form, there is a sequence $\{v_n \in F_n\}$ converging to v_0. We claim that $v_n \in A_n$ for all but finitely many n.

In proving this last claim, we assume that w_0 (used in the labeling) lies in A_0, that v_0 is neither a_0, w_0, nor b_0, and that all prime ends come equipped with representing embeddings starting at u_0. It is not too hard to show that for sufficiently large n, we can assume that terminal portions of the images of representing embeddings for a_0, b_0, v_0, and w_0 are disjoint. For sufficiently large n, we construct a simply connected subdomain G_n' of G_n using

(i) terminal portions of the embeddings for a_n and b_n,

(ii) for α sufficiently close to 1, that portion of the standard simple closed curve $J(\alpha)$ not intersecting the embeddings for w_n or v_n and terminating in intersections with the embeddings for a_n and b_n, and

(iii) the H-interval A_n.

This construction uses the uniform convergence of the representing embeddings. Because $w_n \in A_n$, we have $u_0 \in G_n'$, since the embedding for w_n starts at u_0 and does not meet $\text{bdry}(G_n')$. Using similar reasoning, we must have $v_n \in A_n$ for sufficiently large n, completing the proof of the claim.

To finish the proof of (1), we know that a sequence $\{v_n \in A_n\}$ must have a subsequence converging to some $v_0 \in F_0$, and as above it is not too hard to show that $v_0 \in A_0$. Similarly, we see that $\{B_n\}$ converges to B_0 in the sense of condition (b).

Choose a prime end c_0 in the interior of A_0 such that, as a point, c_0 is not an endpoint of F_0, i.e., c_0 does not have arbitrarily small neighborhoods whose boundaries intersect F_0 in a single point [26, p. 64]. (The non endpoints are dense in F_0.)

For each $\epsilon > 0$, let $V(\epsilon)$ denote the open disk centered at c_0 with radius ϵ, and let $W_0(\epsilon)$ denote the component of $V(\epsilon)\backslash F_0$ whose points can be connected to c_0 with an arc in $V(\epsilon)\backslash F_0$ representing c_0 as a prime end.

(2) There is an $\epsilon_1 > 0$ such that for all ϵ satisfying $0 < \epsilon \leqslant \epsilon_1$, any neighborhood of c_0 inside $V(\epsilon)$ has boundary intersecting F_0 in more than one point.

It is clear that (2) is true because c_0 is not an endpoint of F_0. Using this same fact, it is not too hard to show (3) below.

(3) For all ϵ satisfying $0 < \epsilon \leqslant \epsilon_1$, there are points arbitrarily close to c_0 outside $\text{cl}\,(W_0(\epsilon))$.

(4) There is an $\epsilon_2 \leqslant \epsilon_1$ such that for all ϵ satisfying $0 < \epsilon \leqslant \epsilon_2$ there is a positive integer $N(\epsilon)$ such that if $z_0 \in W_0(\epsilon) \cap V(\epsilon/2)$ and $n \geqslant N(\epsilon)$, then any prime end of F_n representable with an arc from z_0 inside $V(\epsilon)$ must lie in A_n. (See Figure 3.)

If (4) were not true, then, starting with a fixed representing embedding e_0 for c_0, we can construct (taking another subsequence) a sequence $\{v_n \in B_n\}$ of prime ends with representing embeddings e_n which agree with e_0 to within $1/n$ of c_0. This sequence clearly converges to c_0, giving a contradiction to (1).

Choose a fixed $z_0 \in W_0(\epsilon_2) \cap V(\epsilon_2/2)$ satisfying (4), and hence also (3). For each $n \geqslant N(\epsilon)$, let $W_n(\epsilon)$ denote the component of $V(\epsilon)\backslash F_n$ containing z_0.

(5) There is an ϵ satisfying $0 < \epsilon < \epsilon_2$, a δ satisfying $0 < \delta \leqslant \epsilon/2$, and a positive integer N such that $z_0 \in W_0(\epsilon) \cap V(\epsilon/2)$, and such that there is a fixed arc R of the circle $C = \{u\colon |u - z_0| = \delta\}$ which is disjoint from $\text{cl}\,(W_n(\epsilon))$ for all $n \geqslant N$. We assume that R makes an angle of $2\pi/k$ for some integer k.

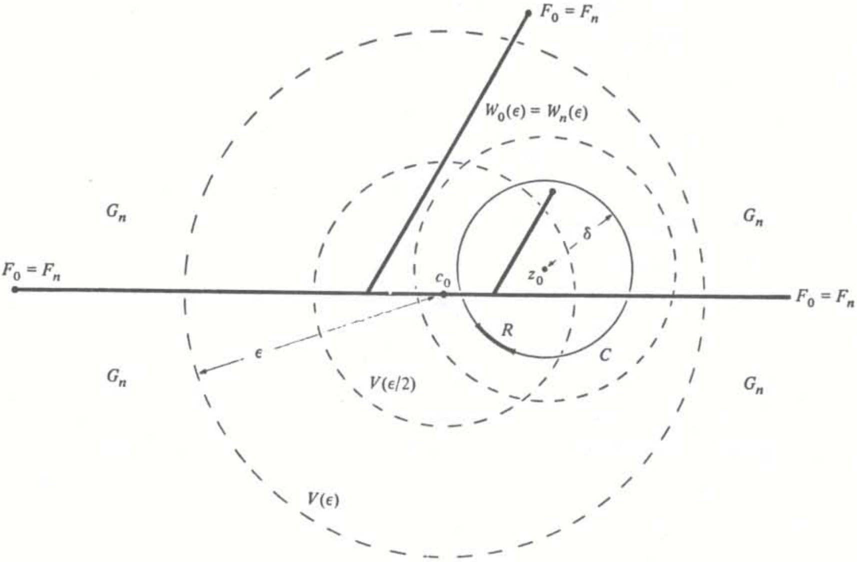

FIGURE 3

Choose $\epsilon < \epsilon_2$ such that $z_0 \in W_0(\epsilon) \cap V(\epsilon/2)$. (It is easy to choose such an ϵ after first choosing an arc from z_0 inside $W_0(\epsilon_2)$ which represents c_0 as a prime end.) Using (3), choose $\delta \leqslant \epsilon/2$ and R such that R is disjoint from $\text{cl}(W_0(\epsilon_2))$. The problem we encounter with (5) is that $W_n(\epsilon)$ may be drastically larger than $W_0(\epsilon)$ for all n. Let $G(\epsilon)$ be the set of all points with a neighborhood contained in $W_n(\epsilon)$ for sufficiently large n. It is clear that $\text{bdry}(W_0(\epsilon))$ is disjoint from $G(\epsilon)$, so the kernel of $\{W_n(\epsilon)\}$ with respect to z_0 (which is the component of $G(\epsilon)$ containing z_0—see Definition 1.3) will be contained in $W_0(\epsilon)$. Now $G(\epsilon)$ may have other components outside $W_0(\epsilon)$, but since $\epsilon < \epsilon_2$, it is not too hard to see that $G(\epsilon)$ will be contained in $W_0(\epsilon_2)$. (By condition (b) and the Carathéodory convergence, a separate component of $G(\epsilon)$ can only occur if portions of F_n extend arbitrarily closely to $\text{bdry}(V(\epsilon))$, for increasing n.) It is now not hard to choose an N so that (5) is true.

Set $g_n(u) = f_n^{-1}(u) - r_0$, for $u \in W_0(\epsilon)$ and $n \geqslant N$. Let m_n be the supremum over all w in $\text{bdry}(W_n(\epsilon))$ of the numbers $\limsup\{|g_n(u)|: u \longrightarrow w, u \in W_n(\epsilon)\}$. By (4), m_n tends to 0 as n tends to infinity. Using (5) and Lemma 1.7, we see that $|g_n(z_0)| \leqslant (m_n 2^{k-1})^{1/k}$. Thus as n tends to infinitely (k fixed), $g_n(z_0)$ tends to zero. On the other hand, $g_n(z_0)$ must converge to $f_0^{-1}(z_0) - r_0$, i.e., $f_0^{-1}(z_0) = r_0$. This is true for infinitely many $z_0 \in G_0$, so we have a contradiction. Thus the set of functions $\{f_n: n \geqslant 1\}$ is equicontinuous on C_1.

The remainder of the proof follows as in [10]. In brief, a subsequence of $\{f_n\}$ converges uniformly on C_1. Hence $\{f_n\}$ converges pointwise on C_1, since otherwise there would be subsequences of $\{f_n\}$ converging to distinct analytic functions on $\operatorname{int}(B^2)$. It follows that $\{f_n\}$ converges uniformly on C_1 and hence on B^2, and clearly it converges to f_0.

PROOF THAT (c) IMPLIES (e). Without loss of generality, we can suppose that each G_n contains the closed unit disk B^2 and that each f_n fixes the origin. For each n, define $g_n: A^2 \longrightarrow A(f_n^{-1}(C_1), C_1)$ as in the proof that (c) implies (e) in Theorem 1.1, and suppose that each g_n maps $(1, 0)$ to $f_n^{-1}(1, 0)$. Note that $h_n = f_n \circ g_n$ fixes the point $(1, 0)$ and that $\{f_n^{-1}(1, 0)\}$ converges to $f_0^{-1}(1, 0)$. By condition (c) it is clear that the sequence of simple closed curves $\{f_n^{-1}(C_1)\}$ Fréchet converges to $f_0^{-1}(C_1)$. Results about mappings of annular regions between simple closed curves show that $\{g_n\}$ converges to g_0 uniformly on A^2. (See [11], [14], [16] and [22].) Thus $\{h_n\}$ converges to h_0 uniformly on A^2. For each n, define a retraction ψ_n of $E^2\backslash\operatorname{int}(B^2)$ onto $E^2\backslash G_n$ as in the proof of Theorem 1.1. Arguing by contradiction, it is not hard to prove that the sequence $\{\psi_n\}$ converges to ψ_0 uniformly on $E^2\backslash\operatorname{int}(B^2)$.

PROOF THAT (e) IMPLIES (d). This is trivial.

PROOF THAT (d) IMPLIES (a). Assume (d) is true and (a) is not true. From (d) it is easy to see that $\{G_n\}$ Carathéodory converges to G_0. If (a) is not true, then part (2) of Definition 2.1 must fail. Thus, for each n there is a cross cut Q_n of G_n such that as n tends to infinity, $\operatorname{diam}(Q_n)$ tends to zero, but $d(G_n\backslash Q_n)$ and $\operatorname{dist}(Q_n, u_0)$ do not tend to zero. The retraction φ_n fixes the endpoints of Q_n, but moves some interior point a distance which does not tend to zero as n tends to infinity. This is clearly a contradiction to the uniform convergence of $\{\varphi_n: n \geqslant 1\}$. Q.E.D.

As in §1, we shall need extensions of Theorem 2.2 to an arbitrary (second countable) 2-manifold M^2. For each nonnegative integer n, let G_n be a precompact simply connected domain in M^2 with locally connected boundary F_n consisting of more than one point. For each n, let $f_n: B^2 \longrightarrow G_n \cup F_n$ be the mapping given by Theorem 1.5, normalized as in the discussion preceding Theorem 1.6. Using the universal covering space as in the proof of Theorem 1.5, we get the following result.

2.4. THEOREM. *The five conditions of Theorem* 2.1 *make sense and are equivalent in this more general setting.*

3. The space of deformation retractions. In this section and the next, we study the space $\mathcal{R}(M^2)$ of retractions of an arbitrary compact 2-manifold M^2, where the compact-open (= sup-metric) topology is used. The continuity property of §2 was developed as a tool for the study of $\mathcal{R}(M^2)$. For material on retractions, see [5] and [13].

There is a number $\epsilon > 0$ such that any two selfmaps of M^2 within a distance ϵ are homotopic. (An argument can be given using geodesics.) Also, it will be clear from work below that the space $\mathcal{D}(M^2)$ of deformation retractions of M^2 is pathwise connected. Hence $\mathcal{D}(M^2)$ is a component in $\mathcal{R}(M^2)$. (See [21] and [22].) We can use Theorem 1.1 to construct a canonical path in $\mathcal{D}(M^2)$ from any deformation retraction to the identity map on M^2, and then Theorem 2.2 implies the following principal result of this section.

3.1. THEOREM. *For any compact 2-manifold M^2, the space $\mathcal{D}(M^2)$ of deformation retractions of M^2 is contractible in itself.*

As a prototype for the construction used in this section and in the proofs of Theorem 4.2 and Lemma 5.3, we have the following, due in its first forms to Alexander [2] and Borsuk [4].

3.2. REMARK [21, p. 320]. Let M be a manifold with boundary ∂M and let $\partial M \times [0, 2] \subset M$ be a collar of the boundary. For $t \in [0, 1]$, let ρ_t be the retraction of M which is the identity outside $\partial M \times [0, t)$ and projects $\partial M \times [0, t]$ to $\partial M \times \{t\}$. Let $h_t: M \longrightarrow \rho_t(M)$ be the homeomorphism given by the identity outside $\partial M \times [0, 2)$ and by mapping $\partial M \times [0, 2]$ linearly to $\partial M \times [t, 2]$. Then the homotopy Θ_t given by $\Theta_t(\varphi) = h_t \circ \varphi \circ h_t^{-1} \circ \rho_t$ provides a deformation of $\mathcal{R}(M)$ in itself such that the image of Θ_1 consists of retractions of M whose images do not meet ∂M.

If M is compact, we can vary the amount a retraction is moved according to the distance it overlaps onto $\partial M \times [0, 1)$. In this way, we can prove the following result.

3.3. THEOREM (ELBOWROOM CONSTRUCTION). *If M is a compact manifold with boundary ∂M and if $\partial M \times [0, 1] \subset M$ is a collar, then there is a strong deformation retraction of $\mathcal{R}(M)$ onto the space of retractions of M with images in $M \backslash (\partial M \times [0, 1))$.*

PROOF OF THEOREM 3.1. The proof is similar to the proof that (c) implies (e) in Theorems 1.1 and 2.2. Let L^2 be the manifold obtained from M^2 by filling in each hole with an open disk. We shall show in Theorem 3.4 below that for any φ in $\mathcal{D}(M^2)$, $L^2 \backslash \mathrm{im}(\varphi)$ consists of a simply connected domain containing each of the open disks. We work in the universal covering space of L^2 (which we assume to be E^2; if it is S^2, the proof is similar), and project the construction down into L^2, handling each hole separately. Thus without loss of generality, we assume that M^2 is $E^2 \backslash \mathrm{int}(B^2)$ and that L^2 is E^2. (This is not compact, but all functions will be fixed outside a bounded portion of E^2.)

Given a deformation retraction φ of $E^2\backslash\text{int}(B^2)$, we can use Remark 3.2 or Theorem 3.3 to assume that $\text{im}(\varphi)$ is disjoint from the boundary (which is C_1 here), and we can assume that $\text{im}(\varphi)$ is bounded in E^2.

First, we show how to produce a canonical deformation retraction ξ_1 of $E^2\backslash\text{int}(B^2)$ onto $E^2\backslash\text{im}(\varphi)$, such that each stage of the deformation is itself a deformation retraction of $E^2\backslash\text{int}(B^2)$. As in the proof of Theorem 1.1, let h be the unique continuous function from $A^2 = A(C_1, C_2)$ onto the closure of $E^2\backslash(\text{im}(\varphi) \cup B^2)$ which is a radical contradiction of a map conformal on the interior and which fixes the point $(1,0) \in C_1$. For each $t \in [0,1]$, let λ_t be the retraction of A^2 onto $A(C_{1+t}, C_2)$ which projects $A(C_1, C_{1+t})$ radially onto C_{1+t}. Define a retraction ξ_t of $E^2\backslash\text{int}(B^2)$ by setting $\xi_t = h \circ \lambda_t \circ h^{-1}$ on the closure of $E^2\backslash(\text{im}(\varphi) \cup B^2)$ and letting ξ_t fix the points of $\text{im}(\varphi)$. For each $t \in [0,1]$, ξ_t is clearly a retraction (it is idempotent) and clearly continuous (the case $t=1$ is contained in the proof of Theorem 1.1). To prove continuity of ξ_t in t when $t=1$, show that, given $\epsilon > 0$, there is a number $r < 1$ such that $\sup\{|\xi_t(u) - \xi_1(u)| : u \in E^2\backslash\text{int}(B^2)\} < \epsilon$ for all t satisfying $r < t \leq 1$. This is not hard to show using the facts that h is uniformly continuous on A^2 and that λ_t moves the points of this annulus radially out towards C_2.

Thus, ξ_t gives a canonical path in the space of deformation retractions of $E^2\backslash\text{int}(B^2)$ from the identity to a canonical retraction ξ_1 of $E^2\backslash\text{int}(B^2)$ onto $\text{im}(\varphi)$. Setting $\sigma_t = \varphi \circ \xi_t$ (a retraction for each t because $\text{im}(\xi_t)$ contains $\text{im}(\varphi)$), we obtain a path from φ to $\xi_1 = \sigma_1$, and ξ_{1-t} gives a path from ξ_1 to the identity on $E^2\backslash\text{int}(B^2)$. Theorem 2.2 shows that these canonical paths give the desired contraction of $\mathcal{D}(E^2\backslash\text{int}(B^2))$. Q.E.D.

The next result characterizes the deformation retracts of M^2 and thus generalizes a theorem of Borsuk [3], [5, p. 132].

3.4. THEOREM. *Let M^2 be any compact 2-manifold and let m be the number of its boundary curves. Let L^2 be a compact 2-manifold without boundary containing disjoint open disks D_j $(1 \leq j \leq m)$ such that $M^2 = L^2\backslash\bigcup\{D_j\}$. Let R be a subset of M^2 such that*

(1) R is connected, locally connected and closed, and

(2) $G = L^2\backslash R$ consists of simply connected components G_j with $D_j \subset G_j$ $(1 \leq j \leq m)$.

Then we can conclude that

(i) R is the image of a deformation retraction ξ of M^2, and

(ii) if R is the image of a retraction φ of M^2, then φ is a deformation retraction, and there is a canonical deformation in $\mathcal{D}(M^2)$ from φ to the identity.

Conversely, let M^2 and L^2 be as above, and let φ be a deformation *retraction of* M^2. *Then* $R = \text{im}(\varphi)$ *satisfies* (1) *and* (2) *above.*

PROOF. If $R \subset M^2$ satisfies (1) and (2) above, then (i), (ii) follow from the proof of Theorem 3.1.

For the converse, let $\varphi \in \mathcal{D}(M^2)$. Condition (1) of the theorem is clear. Suppose the universal covering space of L^2 is E^2, with covering projection p, and let N^2 denote the plane less the infinite collection of disjoint open disks $p^{-1}(D_j)$, for $1 \leqslant j \leqslant m$. The space N^2 is a covering space of M^2, and since φ is a deformation retraction, it will lift to a deformation retraction ψ of N^2 [19].

First, suppose $G = L^2 \backslash R$ contains a component H which is not simply connected. Then we can produce a simple closed curve J in H which is not contractible in H, hence also not contractible in L^2, since $\text{im}(\varphi)$ is connected. Thus $p^{-1}(J)$ will consist of an infinite collection of closed embeddings of E^1 in E^2, each of which separates E^2. This contradicts the fact that $\text{im}(\psi)$ is connected. Hence each component of G is simply connected.

Next, suppose H is a component of G which does not contain any of the disks D_j. Let H' be a component of $p^{-1}(H)$. Since H is simply connected, H' will be open and bounded. The retraction ψ, when restricted to $\text{cl}(H')$, would map $\text{cl}(H')$ into $E^2 \backslash H'$ and fix the points on the boundary. Using the Brouwer fixed point theorem [5, p. 12], it is easy to see that there can be no such map.

Finally, if any component of G contains more than one of the disks D_j, then we could use the first part of this theorem to construct a deformation retraction of L^2 with k holes ($k < m$) onto R. Thus L^2 with k holes would have the same homotopy type as L^2 with m holes, which is not true.

The case in which the universal covering space of L^2 is S^2 can be handled as above by embedding L^2 minus a hole in E^2 and arguing separately if $m = 0$. Q.E.D.

4. **The space of nullhomotopic retractions.** The nullhomotopic retractions of a compact 2-manifold M^2 are exactly those with contractible image, and the space of these retractions, with the compact-open (= sup-metric) topology, is a component in the space $\mathcal{R}(M^2)$ of all retractions of M^2. (As noted in [22, pp. 611–612], this follows because any two sufficiently close selfmaps of M^2 must be homotopic and because the space of nullhomotopic retractions is pathwise connected, as will be shown below.)

4.1. DEFINITION. Let Λ be the canonical embedding of M^2 into its space of retractions which maps each point of M^2 to the constant retraction to that point. Denote by $\mathcal{L}(M^2)$ the component containing the image of Λ, and restrict the range of Λ to $\mathcal{L}(M^2)$. Finally, let ev: $\mathcal{L}(M^2) \longrightarrow M^2$ be the evaluation map which takes a retraction to the retraction evaluated at a basepoint of M^2.

It is clear that $L(M^2)$ is exactly the space of nullhomotopic retractions of M^2.

4.2. THEOREM. *For any compact 2-manifold M^2, the embedding Λ: $M^2 \longrightarrow L(M^2)$ is a homotopy equivalence with homotopy inverse the evaluation map ev. Thus, the space $L(M^2)$ of nullhomotopic retractions of M^2 has the same homotopy type as M^2.*

In outline, the proof is similar to the proofs in [21] and [22], except that here instead of Michael's selection theorem, we use the following result, which will also be needed in §6.

4.3. LEMMA (EXISTENCE OF CROSS SECTIONS [20, p. 55]). *Let (X, τ, Y) be a locally trivial fibre space* [20, p. 3], [19, p. 90] *such that Y is separable metric and the fibre F is an absolute retract* [13, p. 95]. *Then any partial cross section defined on a (possibly empty) closed subspace A of Y extends to a cross section, i.e., a map e: $Y \longrightarrow X$ such that $\tau \circ e$ is the identity on Y.*

Using Theorem 3.3, we can assume that each retraction in $L(M^2)$ has image missing ∂M^2, and this assumption will hold for the rest of this section. Let S denote the collection of all simple closed curves J in M^2 which bound a closed disk $B(J)$ in M^2 and which do not meet ∂M^2. (The disk $B(J)$ is uniquely specified unless M^2 is the 2-sphere.) The set S, with the Fréchet topology (Definition 2.1), becomes a separable metric space [22], and using the methods of [14], it is not hard to see that S is an absolute neighborhood retract (ANR). (In [14, Lemma 11], it is shown that the space of embeddings of B^2 into E^2 which are analytic on $\text{int}(B^2)$ is an ANR. This space has as a retract the subspace consisting of embeddings fixing the origin and having positive derivative at the origin. The space S is locally homeomorphic to the latter space.)

For any $\varphi \in L$, let $S(\varphi)$ denote the space of all $J \in S$ such that $\text{im}(\varphi)$ is contained in $\text{int}(B(J))$, the interior of the disk which J bounds. Each $S(\varphi)$ is nonempty [22, Lemma 1], is contractible in itself [22, Lemma 2(b)], and is clearly open in S. Hence [13, p. 96] $S(\varphi)$ is an absolute retract for each φ. Let B be the subspace of $L \times S$ consisting of all (φ, J) such that $J \in S(\varphi)$, and let τ: $B \longrightarrow L$ be the restriction of the projection onto the first coordinate.

4.4. LEMMA. *The triple (B, τ, L) is a locally trivial fibre space with fibre the absolute retract $S(\Lambda(u_0))$, where u_0 is a basepoint of M^2.*

PROOF. Let $\varphi_0 \in L$. Let U be a neighborhood of φ_0 in L small enough so that for fixed $J_0 \in S$ and some $\epsilon_0 > 0$, we have, for all $\varphi \in U$,

$\text{im}(\varphi) \subset B(J_0)$ and $\text{dist}(\text{im}(\varphi), J_0) > \epsilon_0$. For each $\varphi \in U$, let g_φ be a homeomorphism of $B(J_0)\backslash\text{im}(\varphi)$ onto $B(J_0)\backslash\text{im}(\varphi_0)$ which is a radial contraction of a map conformal on the interior. It is easy to adjust g_φ to a unique homeomorphism (still termed g_φ) which is the identity on J_0. (See [21] and [16, p. 45].) Then extend g_φ to $M^2\backslash\text{im}(\varphi)$ by the identity map. The map g_φ induces a homeomorphism of $S(\varphi)$ onto $S(\varphi_0)$, and this in turn induces a homeomorphism h of $\tau^{-1}(U)$ onto $U \times S(\varphi_0)$, given by $h(\varphi, K) = (\varphi, g_\varphi(K))$. The continuity of h and h^{-1} follows from Theorem 2.2, using the technique in the proof that (c) implies (e). Because L is pathwise connected (as will follow from work below), it is clear that the fibres are all homeomorphic to $S(\Lambda(u_0))$. Q.E.D.

PROOF OF THEOREM 4.2. By Lemma 4.3 (with A empty), there is a cross section $e: L \longrightarrow B$, and if τ_2 is the projection onto the second coordinate, then for each $\varphi \in L$, $\tau_2 \circ e(\varphi)$ is a canonical simple closed curve bounding a disk which contains $\text{im}(\varphi)$ in its interior. (We are still assuming that $\text{im}(\varphi)$ misses ∂M^2, for each $\varphi \in L$.)

The rest of the proof is similar to that in [22, p. 612], but we give a sketch here. For each $\varphi \in L$, let $f_\varphi: B(\tau_2 \circ e(\varphi)) \longrightarrow B^2$ be a conformal map which takes $\varphi(u_0)$ to the origin. The map f_φ is determined only up to a rotation and possibly a reflection (if M^2 is not orientable), but the construction which follows is independent of rotations and reflections. The map $f_\varphi \circ \varphi \circ f_\varphi^{-1}$ is a retraction of B^2, and thus [21, Theorem 1.1] is homotopic to the constant retraction to the origin. The homotopy is similar to that in Remark 3.2, and is given by

$$\psi_t = h_t \circ f_\varphi \circ \varphi \circ f_\varphi^{-1} \circ h_t^{-1} \circ \rho_t,$$

where ρ_t projects $A(C_{1-t}, C_1)$ radially to C_{1-t}, and h_t is a radial homeomorphism of B^2 onto $\rho_t(B^2)$. We define a retraction φ_t piecewise on M^2 for each t. Set

$$\varphi_t(x) = f_\varphi^{-1} \circ \psi_t \circ f_\varphi(x), \quad \text{for} \quad x \in B(\tau_2 \circ e(\varphi))$$

and

$$\varphi_t(x) = f_\varphi^{-1} \circ h_t \circ f_\varphi \circ \varphi(x), \quad \text{for} \quad x \in M^2\backslash\text{int}(B(\tau_2 \circ e(\varphi))).$$

These definitions agree on the intersections of the domains, which is the curve $\tau_2 \circ e(\varphi)$, so together they define a homotopy from φ to $\Lambda(f_\varphi^{-1}(0, 0)) = \Lambda(\varphi(u_0)) = \Lambda \circ ev(\varphi)$. Theorem 2.2 shows that this defines a homotopy from the identity map on L to the map $\Lambda \circ ev$. Clearly $ev \circ \Lambda$ equals the identity map on M^2. Q.E.D.

4.5. REMARK. As in [22], Theorem 4.2 generalizes to an arbitrary (second countable) 2-manifold M^2, provided M^2 is given a metric in which it is complete, and provided we use the sup-metric topology on the space of retractions of M^2.

5. **The space of retracts.** In this section we consider the set of retracts (= images of retractions) of a compact 2-manifold M^2.

5.1. DEFINITION. Let $\mathbf{R}(M^2)$, $\mathbf{D}(M^2)$, and $\mathbf{L}(M^2)$ denote, respectively, the set of retracts of M^2, deformation retracts of M^2, and retracts of M^2 which are images of elements in $\mathcal{L}(M^2)$. (Thus $\mathbf{L}(M^2)$ consists of the set of compact absolute retracts in M^2 [13].) We give $\mathbf{R}(M^2)$ the quotient topology determined by the natural projection im: $\mathcal{R}(M^2) \longrightarrow \mathbf{R}(M^2)$ which maps a retraction to its image. The sets $\mathbf{D}(M^2)$ and $\mathbf{L}(M^2)$ are given topologies as subspaces of $\mathbf{R}(M^2)$.

5.2. REMARK. This topology on $\mathbf{R}(M^2)$ is strictly larger (= more open sets) than that of the Hausdorff metric. This follows because Fréchet convergence implies convergence in the Hausdorff metric, while simple examples show that the reverse implication is not true. (Since our retracts are compact, convergence in the Hausdorff metric is the same as the concept of "convergence" in [26, p. 10], i.e., the limit superior and the limit inferior are equal. See [12] and [17, pp. 50–58].)

For the statement of the following useful lemma, note that we define a *submanifold* of M^2 to be a compact subset which is a 2-manifold.

5.3. LEMMA. *A subset $R \subset M^2$ is a retract of M^2 if and only if there is a submanifold N^2 of M^2 such that*

(i) *R is a deformation retract of N^2, and*

(ii) *N^2 is a retract of M^2.*

PROOF. If (i) and (ii) hold, then clearly R is a retract of M^2. Suppose that R is a retract of M^2. For now, we shall assume that R is disjoint from ∂M^2. Given $u_0 \in \mathrm{bdry}(R)$ and given an embedding $e_0: I = [0, 1] \longrightarrow M^2$ representing u_0 as a prime end (of the first kind, since R is locally connected), we can construct a continuous function $E: I \times I \longrightarrow M^2$ with the following properties:

(1) $E(t, 0) = e_0(t)$, for all $t \in I$,

(2) for each $t \in I$, $E(\cdot\,, t)$ is an embedding representing a prime end of R, and

(3) E restricted to $[0, 1) \times I$ is an embedding into $M^2 \backslash R$.

Two functions E and E' satisfying (2) and (3) *overlap properly* if exactly one of $E(t, 1) = E'(t, 0)$ or $E(t, 0) = E'(t, 1)$ holds, for all $t \in I$, and otherwise the images under E and E' of $[0, 1) \times I$ are disjoint.

We now construct finitely many mappings $E_1, E_2, \cdots, E_n$ satisfying (2) and (3) above and satisfying

(4) $E_i([0, 1) \times (0, 1))$ is disjoint from $E_j([0, 1) \times (0, 1))$ unless $i = j$,

(5) each E_i overlaps properly with exactly two others (in the two ways above), and

(6) each prime end of $M^2 \backslash R$ is represented by $E_i(\cdot, t)$ for some i and t.

The compactness and local connectedness of $\text{bdry}(R)$ insure that this construction may be carried out. Because R is an ANR, we see that $M^2 \backslash R$ can only have finitely many components. (We are still assuming that R is disjoint from ∂M^2.) This fact is proved for the case $M^2 = E^2$ in [5, p. 138]. It follows that R and the images of the E_i must form a compact 2-manifold with boundary the union of the arcs $E_i(0, 1)$. Using the technique in the proof of Theorem 3.1, it is easy to produce a deformation retraction from this manifold onto R.

In order to prove condition (ii) of the lemma, we define the 2-manifold N^2 to be $N^2 = R \cup (\bigcup_{i=1}^n E_i([2/3, 1) \times I))$. Let φ be a retraction of M^2 with image R. From φ, we shall construct a retraction ψ of M^2 with image N^2.

In the above work, we have constructed a half-open collar $[0, 1) \times \partial N^2$ given by a continuous function $p: I \times \partial N^2 \longrightarrow M^2$ which is a homeomorphism except on $\{1\} \times \partial N^2$, where it provides a one-to-one correspondence between points and prime ends of $M^2 \backslash R$. In this notation, N^2 is $R \cup ([2/3, 1) \times \partial N^2)$ and ∂N^2 is the same as $\{2/3\} \times \partial N^2$.

We define the retraction ψ piecewise on the following four closed subsets of M^2: (i) N^2, (ii) $[1/3, 2/3] \times \partial N^2$, (iii) $[0, 1/3] \times \partial N^2$, and (iv) $M^2 \backslash (N^2 \cup (0, 2/3) \times \partial N^2)$. On N^2, we take ψ equal to the identity map, and on the subset in (iv), we set $\psi = \varphi$. For the subset in (ii), we map $[1/3, 2/3]$ linearly onto $[2/3, 1]$ (sending $1/3$ to 1 and fixing $2/3$) and project using p, so that $[1/3, 2/3] \times \partial N^2$ is projected by ψ onto $([2/3, 1) \times \partial N^2) \cup \text{bdry}(R)$, which is a subset of N^2. For the subset in (iii), map $[0, 1/3]$ linearly onto $[0, 1]$ (fixing 0 and sending $1/3$ to 1), project by p into M^2 and follow by φ suitably restricted. (Thus ψ takes $\{1/3\} \times \partial N^2$ onto $\text{bdry}(R)$ and $\{0\} \times \partial N^2$ into R by the map φ.) It is clear that these definitions agree on the intersections of the domains, so together they give a continuous map ψ, which is also clearly a retraction with image N^2. (See Figure 4.)

If R meets ∂M^2, attach a collar to ∂M^2 and carry out the above construction in such a way that if a point $(t, u) \in I \times \partial N^2$ belongs to M^2, then the whole interval $[t, 1) \times \{u\}$ belongs to M^2. We omit the details. Q.E.D.

The following theorem shows that the quotient topology of Definition 5.1 is natural in other ways. Notice the similarities with Theorems 1.1 and 2.2. In the statement below, for each submanifold N^2 of M^2, let L^2 be the manifold obtained by filling in each hole of N^2 with an open disk.

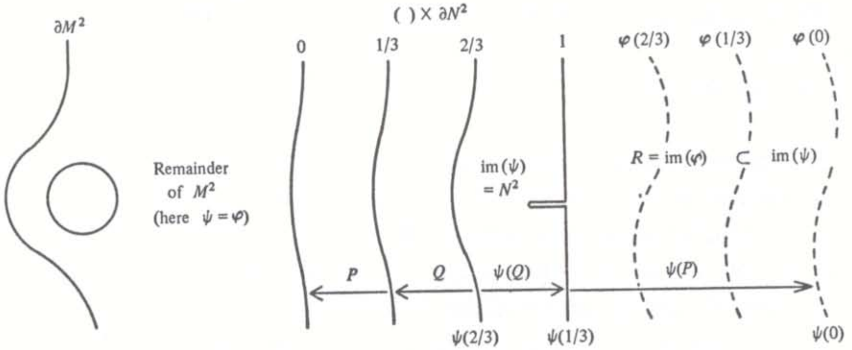

FIGURE 4

5.4. THEOREM (CONTINUITY PROPERTY FOR RETRACTS). *Let M^2 be a compact 2-manifold, and for each nonnegative integer n, let R_n be a retract of M^2. Then the following are equivalent.*

(a) *The sequence $\{R_n: n \geqslant 1\}$ converges to R_0 in the topology of Definition* 5.1.

(b) *There is a submanifold N^2 of M^2 which is a retract of M^2 and an integer n_0 such that for $n = 0, n \geqslant n_0$, N^2 contains R_n and $L^2 \backslash R_n$ consists of simply connected domains, a domain G_n for each hole of N^2, and for each hole, $\{\mathrm{bdry}(G_n): n \geqslant n_0\}$ Fréchet converges to $\mathrm{bdry}(G_0)$.*

(c) *With N^2, n_0, L^2 and G_n as in* (b), *if $f_n: B^2 \longrightarrow \mathrm{cl}(G_n)$ is the corresponding map conformal on $\mathrm{int}(B^2)$ (normalized as in the discussion before Remark* 1.6), *then for each hole of N^2, $\{f_n: n \geqslant n_0\}$ converges to f_0 uniformly on B^2.*

(d) *For each n, there is a retraction φ_n of M^2 with image R_n such that $\{\varphi_n: n \geqslant 1\}$ converges to φ_0 uniformly on M^2.*

(e) *There is a submanifold N^2 of M^2 which is a retract of M^2 and an integer n_0 such that for $n = 0, n \geqslant n_0$, there are deformation retractions ψ_n of N^2 with image R_n such that $\{\psi_n: n \geqslant n_0\}$ converges to ψ_0 uniformly on N^2.*

PROOF. It is clear from the definition of quotient topology that (a) is equivalent to (d). The fact that (b) implies (c) is clear from Theorem 2.2. The proof that (c) implies (e) can be carried out using the construction in the proof of Theorem 3.1. Assuming (e), Theorem 3.4 shows that $L^2 \backslash R_n$ consists of a simply connected domain G_n for each hole of N^2. Since each ψ_n ($n = 0$, $n \geqslant n_0$) is a deformation retraction of N^2, these maps lift to the covering space of N^2 given by taking the universal covering space of L^2 and omitting the lifted holes. This will be the plane with holes, and now Theorem 2.2 implies the Fréchet convergence of $\{\mathrm{bdry}(G_n): n \geqslant n_0\}$ to $\mathrm{bdry}(G_0)$ for each hole of N^2. Hence we get (b). It remains to show that (d) and (e) are equivalent.

Assuming (e) again, let $\varphi: M^2 \longrightarrow N^2$ be a retraction and set $\varphi_n = \psi_n \circ \varphi$. It is clear that (d) holds.

Finally, assuming (d), construct N^2 from R_0 as in the proof of Theorem 5.3. Choose ϵ less than $\mathrm{dist}(R_0, M^2\backslash N^2)$ such that any two selfmaps of N^2 within a distance ϵ are homotopic in N^2. Choose n_0 such that $n \geqslant n_0$ implies φ_n is within ϵ of φ_0. Thus for all $n \geqslant n_0$, $R_n \subset N^2$ and $\varphi_n|_{N^2}$ is homotopic to $\varphi_0|_{N^2}$ in N^2. Since R_0 is a deformation retract of N^2 and $\varphi_0|_{N^2}$ is a retraction, we see by Theorem 3.4 that $\varphi_0|_{N^2}$ is a deformation retraction. Hence $\varphi_n|_{N^2}$ is a deformation retraction of N^2 for $n \geqslant n_0$. Setting $\psi_n = \varphi_n|_{N^2}$, we get (e). Q.E.D.

5.5. THEOREM. *For any compact 2-manifold M^2, (i) the space $\mathbf{R}(M^2)$ of retracts of M^2 is an ANR (for metrizable spaces [13, Chapter 3]),*

(ii) *The space $\mathbf{D}(M^2)$ of deformation retracts of M^2 is an AR, and*

(iii) *The space $\mathbf{L}(M^2)$ of compact AR subsets of M^2 is an ANR with a homeomorphic copy of M^2, $\mathrm{im}(\Lambda(M^2))$, as a deformation retract, and thus $\mathbf{L}(M^2)$ has the same homotopy type as M^2.*

PROOF. Let R be a retract of M^2 and choose N^2 as in Theorem 5.4. Suppose $L^2 = N^2 \cup (\bigcup \{D_j: 1 \leqslant j \leqslant m\})$ and $L^2\backslash R = \bigcup\{G_j: 1 \leqslant j \leqslant m\}$ are as in Theorem 3.4. If R is disjoint from $\mathrm{bdry}(D_j)$ for all j and if $\mathrm{bdry}(G_j)$ is disjoint from $\mathrm{bdry}(G_k)$ for $j \neq k$, then at the member R, the space $\mathbf{R}(M^2)$ is locally homeomorphic to a product of copies of the ANR $\mathrm{A}(B^2, E^2)$ described below (one copy for each hole of N^2), and hence [13, p. 97] is locally an ANR at R. The rest of the proof is needed to take care of the fact that R may not be nicely situated in M^2.

(1) Let $\mathrm{A}'(B^2, E^2)$ denote the space of maps of B^2 into E^2 which are conformal homeomorphisms on $\mathrm{int}(B^2)$. This space was shown to be an ANR in [14, Lemma 9].

(2) Let $\mathrm{A}(B^2, E^2)$ denote the subspace of $\mathrm{A}'(B^2, E^2)$ consisting of maps which fix the origin and have positive derivative at the origin. This space is clearly a retract of $\mathrm{A}'(B^2, E^2)$, and hence [13, p. 97] is an ANR.

For L^2 as above (without boundary) and any $u_1 \in L^2$, let $\mathrm{A}(B^2, L^2)\cdot$ denote the space of maps of B^2 into L^2 which send the origin to u_1, are conformal homeomorphisms on $\mathrm{int}(B^2)$, and send the positive x-axis to a fixed direction at u_1.

(3) The space $\mathsf{A}(B^2, L^2)$ is an r-image of $\mathsf{A}(B^2, E^2)$ and hence [5, p. 87] is an ANR. Also it follows from Theorem 2.2 that $\mathsf{A}(B^2, L^2)$ is homeomorphic to the space $\mathbf{D}_0(L^2\backslash \{u_1\})$ of those deformation retracts of $L^2\backslash\{u_1\}$ whose images miss some neighborhood of u_1.

To prove (3), we first describe a canonical *shrinking* of a map $f \in \mathsf{A}(B^2, E^2)$ by a *factor* β $(0 < \beta \leqslant 1)$. Let g_β be the obvious radial homeomorphism of B^2 onto $\beta \cdot B^2$ and set $\beta \cdot f\colon B^2 \longrightarrow L^2$ equal to $(f|_{\beta \cdot B^2}) \circ g_\beta$. (Notice that for $\beta < 1$, the image of $\beta \cdot f$ is a proper subset of the image of $\operatorname{int}(B^2)$ under f. Notice also that $\beta \cdot f \in \mathsf{A}(B^2, E^2)$.) Let $p\colon E^2 \longrightarrow L^2$ be a conformal universal covering projection such that p maps the origin to u_1 and the positive x-axis to the fixed direction at u_1. (The proof is similar if the universal covering space of L^2 is S^2 or $\operatorname{int}(B^2)$.) We now define an r-map from $\mathsf{A}(B^2, E^2)$ onto $\mathsf{A}(B^2, L^2)$ by sending f to $p \circ (\beta \cdot f)$, where we take the largest value of β for which the image map belongs to $\mathsf{A}(B^2, L^2)$. (In order to construct a right inverse to our r-map, we need the facts that a simply connected domain in L^2 will lift under p to homeomorphic copies, and that the functions in $\mathsf{A}(B^2, E^2)$ and $\mathsf{A}(B^2, L^2)$ are uniquely determined by the image of $\operatorname{int}(B^2)$.)

For distinct points $u_1, u_2, \cdots, u_m \in L^2$, define $\mathsf{A}^{(m)}$ to be the space of all m-tuples $(f_1, f_2, \cdots, f_m)$ such that each f_j belongs to $\mathsf{A}(B^2, L^2)$ (sending the origin to u_j) and such that $f_j(\operatorname{int}(B^2))$ is disjoint from $f_k(\operatorname{int}(B^2))$ for $j \neq k$.

(4) $\mathsf{A}^{(m)}$ is a retract of the product of m copies of the ANR $\mathsf{A}(B^2, L^2)$, and hence [13, p. 97] is an ANR. (This can be proved using a shrinking procedure similar to that described after (3) for maps in $\mathsf{A}(B^2, L^2)$.)

(5) The space $\mathbf{D}(N^2)$ is a retract of the space

$$\mathbf{D}_0(L^2\backslash\{u_1, u_2, \cdots, u_m\})$$

consisting of deformation retracts missing a neighborhood of each u_j. (Here $u_j \in D_j$ for each j.) The latter space is homeomorphic to $\mathsf{A}^{(m)}$, and hence $\mathbf{D}(N^2)$ is an ANR.

The retraction in (5) is not hard to construct using the method in the proof of Theorem 3.3. Here we use Remark 3.2 to move a retract a varying distance depending on how far it overlaps onto the disks D_j.

Finally, Theorem 5.4 shows that the space $\mathbf{R}(M^2)$ is locally homeomorphic to $\mathbf{D}(N^2)$ (for varying N^2), and hence [13, p. 98] $\mathbf{R}(M^2)$ is an ANR. Parts (ii) and (iii) of the theorem follow from corresponding statements about $\mathcal{D}(M^2)$ and $\mathsf{L}(M^2)$ (Theorems 3.1 and 4.2). Q.E.D.

6. **Global forms of the continuity property.** In Theorem 3.1, Lemma 4.4, and the proof of Theorem 5.5, we encountered global forms of Theorem 2.2.

There are other similar results, and in this section we present two which seem interesting. Throughout, we assume that M^2 is an orientable Riemann surface [1] with a continuous nonvanishing vector field. Thus, M^2 must be noncompact, or with nonempty boundary, or must be the torus. (See [6] for other applications of vector fields on 2-manifolds.)

Let $\mathcal{F}$ be the collection of all locally connected continua in M^2 such that each $F \in \mathcal{F}$ is the boundary of a simply connected domain G_F, and give $\mathcal{F}$ the topology of Fréchet convergence (Definition 2.1). The space $\mathcal{F}$ is locally homeomorphic to a G_δ subset of a separable Banach space [14, pp. 278–279] and hence is locally complete. As in [22, Lemma 2(a)], we see that $\mathcal{F}$ is a topologically complete separable metric space. The results which follow also hold for subspaces of $\mathcal{F}$; for example, the space of simple closed curves bounding a disk.

Let $\mathcal{G}$ be the subspace of $\mathcal{F} \times M^2$ consisting of all (F, u) such that $u \in G_F$, and let $\tau\colon \mathcal{G} \longrightarrow \mathcal{F}$ be the restriction of the projection onto the first coordinate. Each fibre $\tau^{-1}(F)$ is homeomorphic to $\operatorname{int}(B^2)$ and hence is an AR. It follows from Theorem 2.2 that $(\mathcal{G}, \tau, \mathcal{F})$ is a locally trivial fibre space. Thus by Lemma 4.3, there is a cross section $e\colon \mathcal{F} \longrightarrow \mathcal{G}$. If τ_2 is the projection onto the second coordinate, then $\tau_2 \circ e\colon \mathcal{F} \longrightarrow M^2$ allows us to select continuously a canonical point u_F from G_F for each $F \in \mathcal{F}$.

6.1. Definition. For each $F \in \mathcal{F}$, define $f_F\colon B^2 \longrightarrow G_F \cup F$ to be the canonical continuous surjection which is a conformal homeomorphism on $\operatorname{int}(B^2)$, which maps the origin to u_F, and which maps the direction of the positive x-axis to the direction given by the vector field.

Let g be the homeomorphism of $\mathcal{F} \times \operatorname{int}(B^2)$ onto $\mathcal{G}$ defined by $g(F, u) = (F, f_F(u))$. Let $\mathcal{A}$ denote the space of all maps $f\colon B^2 \longrightarrow M^2$ which are conformal homeomorphisms on $\operatorname{int}(B^2)$, and let $p\colon \mathcal{A} \longrightarrow \mathcal{F}$ be the natural continuous projection which maps f to the image of C_1 under f. We can now easily derive the next result, which might be called a "global form of the Schoenflies theorem."

6.2. Theorem. *Using notation above,*

(i) *The homeomorphism g satisfies $\tau \circ g = \tau_1$, the projection onto the first coordinate, i.e., $(\mathcal{G}, \tau, \mathcal{F})$ is equivalent to the trivial fibre space $(\mathcal{F} \times \operatorname{int}(B^2), \tau_1, \mathcal{F})$ and*

(ii) *the injection from $\mathcal{F}$ into $\mathcal{A}$ which maps F to f_F is a continuous right inverse for the projection p.*

One can see from Theorem 2.2 that the solutions to the Dirichlet problem on a Riemann surface for simply connected domains with locally connected boundaries change continuously if the boundary values change continuously and the boundaries change continuously in the topology of Fréchet convergence (see [7]). For a global statement of the above, we need the vector field. Let $\mathcal{F}$, G_F, and f_F be defined as above and let $\overline{\mathcal{G}}$ be the set of all $(F, u) \in \mathcal{F} \times M^2$ such that $u \in G_F \cup F$. For each $F \in \mathcal{F}$, suppose there is given a continuous function $b_F: F \longrightarrow E^1$ such that the function B defined on $\mathcal{F}$ by $B(F) = b_F \circ (f_F|_{C_1})$ is continuous, where the range of B is the space of continuous real-valued functions on C_1 with the sup-metric topology. (For example, one could use any continuous real-valued function b defined on the union of the members of $\mathcal{F}$ and set $b_F = b|_F$.) We now have a global form of a solution to the Dirichlet problem.

6.3. THEOREM. *Using notation above, there is a unique continuous real-valued function h defined on $\overline{\mathcal{G}}$ such that for fixed $F \in \mathcal{F}, h(F, \cdot)$ is harmonic on G_F and agrees with b_F on F.*

PROOF. For each $F \in \mathcal{F}$, let $k_F: B^2 \longrightarrow E^1$ be continuous on B^2, harmonic on $\operatorname{int}(B^2)$ and agree with $b_F \circ (f_F|_{C_1})$ on C_1. Then $k_F \circ f_F^{-1} = h_F$ is harmonic on G_F, agrees with b_F on F, and is continuous on $G_F \cup F$. We define $h(F, u) = h_F(u)$, for $F \in \mathcal{F}$ and $u \in G_F \cup F$. Arguing by contradiction, we see that h is continuous. Q.E.D.

ADDED IN PROOF. Robert Cauty of Paris, France has independently obtained the results of §4 using completely different methods.

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