Constructions with Pentacubes

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As introduced in [1, p. 118, p. 159], the *pentacubes* consist of the twenty-nine distinct ways (including reflections) of connecting five unit cubes together on their faces. (Twelve of these are the standard solid pentominoes, one unit high.) Eliminating the "straight" pentacube, we obtain the twenty-eight non-convex pentacubes which we shall work with.

The purpose of this note is to indicate the ease with which pentacube constructions may be carried out using modules which can be combined in various ways. We give three examples, chosen from a large number of similar constructions which the author has carried out. All three are relatively easy to construct in many different ways. The figures show the various levels, where the bottom level is numbered 1, and a dot represents a cube attached above.

Figure 1 shows four modules which can be combined to yield a cylinder seven cubes high with cross section a double-sized model of any of the twelve pentominoes, thus giving at once all twelve constructions in [1, Problem 56, p. 160]. (In three of the constructions, the "N" and "Y" pentacubes of Module a must be rotated.) A number of other cylinders can be constructed with these modules.

Figure 2 illustrates five modules which will yield triple-sized models of eighteen of the pentacubes (the "linear" ones), with one pentacube left over [1, Problem 55].

Finally, Figure 3 gives six modules which will form a wide variety of cylinders five cubes high. It would be interesting to see a construction of seven modules of the type in Figure 3, i.e., cylinders five units high. It is not hard to see that a construction of eight or more such modules is impossible, since this would involve producing four or more modules containing three pentacubes each, and at most three such modules can be produced simultaneously.

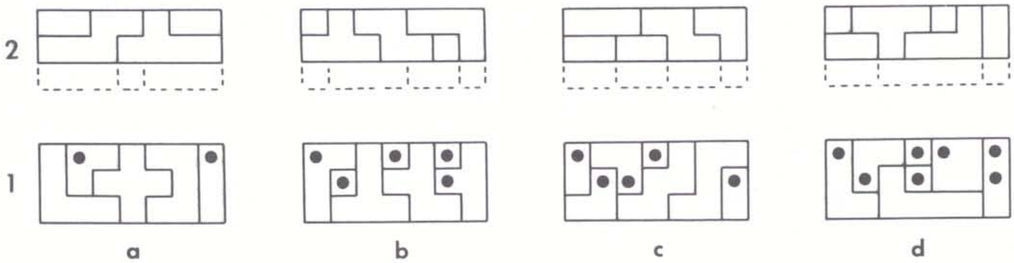

FIGURE 1

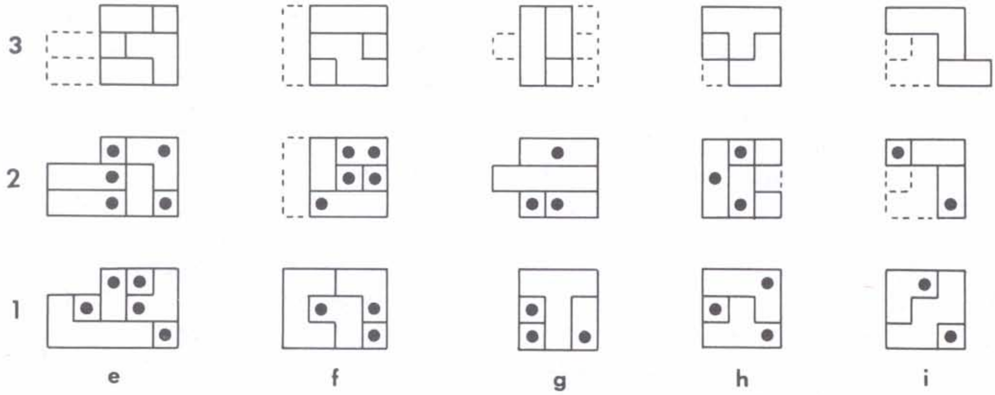

FIGURE 2

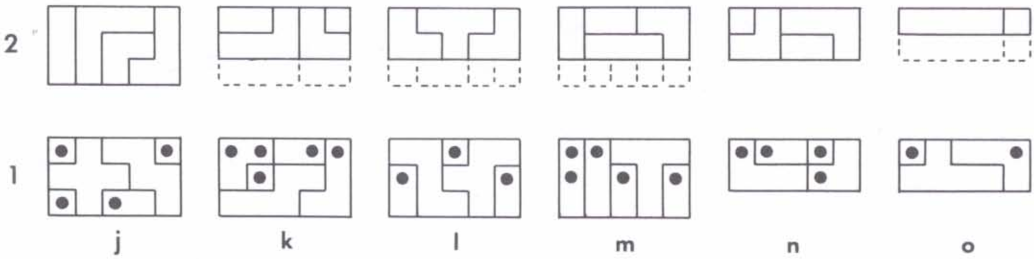

FIGURE 3

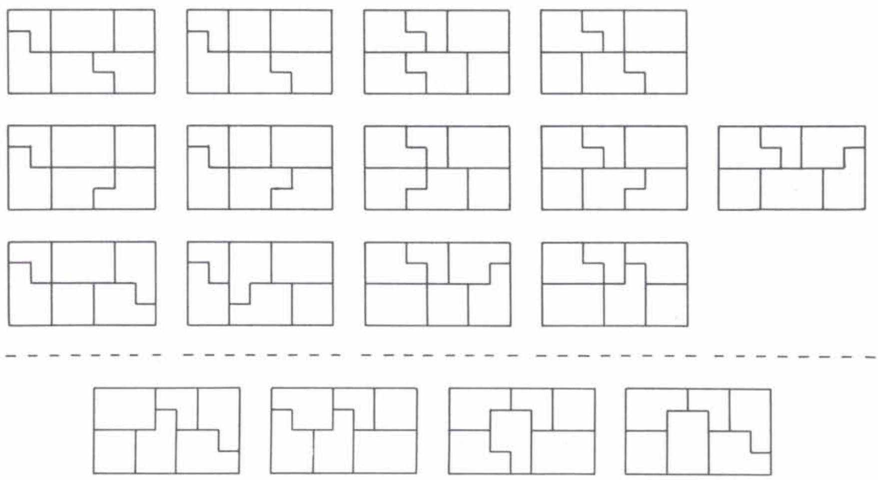

FIGURE 4

A fairly straightforward count shows that there are $2^{10} \cdot 3 \cdot 7 = 21{,}504$ different ways to construct a 4 x 5 x 7 box using the modules in Figure 3, where two solutions differing by a rotation or reflection of the box are not considered distinct. There are 56 basic types of solutions, corresponding to the different ways of assembling cross sections of the modules into a 4 x 7 rectangle (not counting rotations and reflections of the rectangle). The first 13 illustrations of Figure 4 list these according to the location of a 2 x 4 rectangle formed from Modules o and k and then according to the location of Module j, where each illustration represents four types (obtained by moving Module o around). Each of the remaining four illustrations in Figure 4 lists a type for which Module o is not located inside a 2 x 4 rectangle. For each of the 56 basic types, there are $4 \cdot 8 \cdot 2 \cdot 6$ different solutions resulting from rotations of Modules j, n, and o, and permutations of Modules k, l, and m. In addition, Module o can be constructed in reflected form to yield another 21,504 solutions. This gives an indication of the very large number of ways in which a simple Figure like a 4 x 5 x 7 box can be constructed.

Reference

1. S. W. Golomb, *Polyominoes,* Charles Scribner's Sons, New York, 1965.