A CONTINUITY PROPERTY AND SURFACE TOPOLOGY

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Communicated by R. D. Anderson, April 10, 1973

A reprint from the

Bulletin

of the

AMERICAN MATHEMATICAL SOCIETY

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Introduction. While studying global properties of spaces of retractions of 2-manifolds, the author was led to a continuity property for conformal mappings of simply connected domains with locally connected boundaries (§1, Theorem 2). Applications are given to the study of spaces of retractions (§2), spaces of retracts (§3), and the Dirichlet problem (§4). Details of the proofs will appear elsewhere.

1. **The continuity property.** For each nonnegative integer i, let G_i be a bounded simply connected domain in the plane E^2 such that each G_i contains some fixed closed disk D with the origin as center. We first state a known result.

THEOREM 1 (CARATHÉODORY, ET AL.; BORSUK). *For each i, the following are equivalent.*

(a) *The boundary F_i of G_i is locally connected.*

(b) *There is a unique continuous surjection f_i from the closed unit disk B^2 to $\text{cl}(G_i) = G_i \cup F_i$ which fixes the origin, is conformal on $\text{int}(B^2)$, and has positive derivative at the origin.*

(c) *There is a retraction of $E^2 \backslash \text{int}(D)$ with image $E^2 \backslash G_i$.*

From [8, p. 112] it is clear that F_i is locally connected if and only if all the prime ends of G_i are of the first kind [4, p. 65], and hence [4, p. 67] (a) implies (b). An elementary construction allows one to prove that (b) implies (c). (Borsuk originally showed directly that (a) implies (c).) It is clear that (b) or (c) implies (a).

By an abuse of language, we consider each point $v \in F_i$ as a prime end of the first kind, i.e., as a point with a representing embedding

$e:[0, 1] \rightarrow G_i \cup F_i$, where $e([0, 1)) \subset G_i$ and $e(1) = v$. A sequence $\{v_i \in F_i : i \geq 1\}$ of prime ends *converges* to a prime end $v_0 \in F_0$ if we can select representing embeddings e_i for v_i such that $\{e_i : i \geq 1\}$ converges uniformly to e_0 on $[0, 1]$.

DEFINITION. The sequence $\{F_i : i \geq 1\}$ *Fréchet converges* to F_0 if (1) each prime end $v_0 \in F_0$ is the limit of a sequence $\{v_i \in F_i : i \geq 1\}$ of prime ends, and (2) each subsequence of a sequence $\{v_i \in F_i : i \geq 1\}$ of prime ends has, in turn, a subsequence which converges to a prime end of F_0.

See [1] for other generalizations of Fréchet convergence (also termed 0-*regular* convergence [3], [7]) and for a result similar to (and more general than) the equivalence of (a) and (b) in Theorem 2 below.

THEOREM 2. *The following are equivalent.*

(a) *The sequence* $\{F_i : i \geq 1\}$ *Fréchet converges to* F_0.

(b) *The sequence* $\{f_i : i \geq 1\}$ *converges uniformly to* f_0 *on the closed disk* B^2.

(c) *There are retractions* ϕ_i *of* $E^2 \backslash \text{int}(D)$ *with image* $E^2 \backslash G_i$ *such that the sequence* $\{\phi_i : i \geq 1\}$ *converges uniformly to* ϕ_0 *on* $E^2 \backslash \text{int}(D)$.

The proof that (a) implies (b) can be carried out with Lindelöf's lemma using a modification of the proof in [2, pp. 59–62]. The proof that (b) implies (c) is similar to the corresponding proof in Theorem 1. A fairly complicated proof shows that the denial of (a) implies the denial of (c).

Using covering spaces, Theorems 1 and 2 generalize to precompact simply connected domains with locally connected boundaries on an arbitrary 2-manifold.

2. The space of retractions.

We shall consider two components of the space of retractions of any compact 2-manifold with the compact-open (= sup-metric) topology.

THEOREM 3. *The space of deformation retractions of* M^2 *is contractible in itself.*

The contraction can be constructed directly using the conformal mappings of Theorem 2(b).

Let Λ be the embedding of M^2 into the space of nullhomotopic retractions of M^2 (= retractions with contractible image) which maps each point to the constant retraction to that point.

THEOREM 4. *The embedding* Λ *is a homotopy equivalence with homotopy inverse the map ev which evaluates a retraction at a basepoint of* M^2.

The proof is similar to the proofs in [6] and [7], except that here we use a cross section theorem [5, p. 55] instead of Michael's selection theorem.

We also use the fact that the space of simple closed curves in M^2 is an ANR [3].

3. **The space of retracts.** The set of retracts (= images of retractions) of any compact 2-manifold M^2 is given a natural quotient topology of the space of retractions of M^2 by saying that a sequence of retracts $\{R_i : i \geq 1\}$ converges to R_0 if there are retractions ϕ_i of M^2 with image R_i ($i \geq 0$) such that $\{\phi_i : i \geq 1\}$ converges uniformly to ϕ_0 on M^2.

THEOREM 5. (i) *The space of retracts of M^2 is an ANR.*

(ii) *The space of deformation retracts of M^2 is an AR.*

(iii) *The space of compact AR subsets of M^2 is an ANR with a homeomorphic copy of M^2 as a deformation retract.*

We use the fact that any retract of M^2 will be a deformation retract of a submanifold N^2 of M^2, where N^2 is a retract of M^2. The space of deformation retracts of N^2 is homeomorphic to a retract of a product of ANR's (one for each hole of N^2), where the ANR's are the space of those maps of B^2 into a 2-manifold L^2 (obtained by filling in the holes of N^2) which send the origin to a fixed $u_0 \in L^2$, are conformal on int(B^2), and send the positive x-axis to a fixed direction at u_0. The latter space is an r-image of a retract of a space studied in [3, Lemma 9].

4. **The Dirichlet problem.** In this section we assume M^2 is an orientable Riemann surface whose interior has a continuous nonvanishing vector field. Let $\mathscr{F}$ be any collection of locally connected continua in M^2 such that each $F \in \mathscr{F}$ is the boundary of a simply connected domain G_F, and give $\mathscr{F}$ the Fréchet topology. We can use the cross-section theorem of [5, p. 55] to produce continuously, for each $F \in \mathscr{F}$, a point $u_F \in G_F$, and we then obtain a canonical continuous function $f_F : B^2 \to G_F \cup F$ which is conformal on int(B^2), takes the origin to u_F, and takes the direction of the positive x-axis to the direction given by the vector field.

For each $F \in \mathscr{F}$, suppose there is given a continuous real-valued function b_F defined on F such that the function B defined on $\mathscr{F}$ by $B(F) = b_F \circ (f_F|_{S^1})$ is continuous, where the range is the space of continuous real-valued functions on S^1 with the sup-metric topology. (For example, one could use any continuous real-valued function b defined on the union of the members of $\mathscr{F}$ and set $b_F = b|_F$.) Let $\mathscr{G} \subset \mathscr{F} \times M^2$ consist of all (F, u) such that $u \in G_F \cup F$.

THEOREM 6. *There is a continuous real-valued function h defined on $\mathscr{G}$ such that for fixed $F \in \mathscr{F}$, $h(F, \cdot)$ is harmonic on G_F and agrees with b_F on F.*

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